Table 3 Influence of noise on damping ratio, %

Mode no.	Percent noise-to-signal ratio			
	0	10	30	50
1	2.040	2.037	2.059	1.974
2	2.054	2.036	2.086	1.949
3	2.004	2.009	2.073	2.110
4	1.989	2.013	2.197	1.759
5	2.016	2.005	1.941	2.150

Table 4 Comparison of results between the sparse time domain algorithm and the least-squares moving-block technique

Mode no.	Frequency, Hz			Damping ratio, %		
		30% N/S ratio			30% N/S ratio	
	Exact	STD	LSMBT	Exact	STD	LSMBT
1	10.00	9.99	10.00	2.00	2.11	2.06
2	12.00	11.99	12.00	2.00	2.05	2.07
3	15.00	15.00	15.00	2.00	2.03	2.07
4	20.00	20.01	20.00	2.00	2.02	2.20
5	21.00	21.03	21.00	2.00	1.99	1.94

To improve the accuracy of the results, scaling factors for the time history data may be used. In particular, scaling factors can be selected in such a way as to force the peak-to-peak amplitude of the first block to be the same for all time signals.

Comparison to the Ibrahim Sparse Time Domain Algorithm

The LSMBT is applied to the numerical test solved by the sparse time domain (STD) algorithm in Ref. 2 using the same modal frequencies, damping ratios, and eigenvectors. The selected time increment, frequency resolution, block length, block step size, and number of blocks are

$$\Delta t = 0.004 \text{ s}, \quad \Delta f = 0.5 \text{ Hz}, \quad N = 100, \quad K_B = 100, \quad l = 3$$

Random numbers with a uniform distribution are added to the time history data for a given noise-to-signal ratio. The influence of noise on the damping ratio estimates is shown in Table 3. Good estimates for the damping ratios are retained up to a high 50% N/S ratio. The effect of noise is governed by the number of sampled data $p \times N \times l$. As statistically expected, a large number of data yield better accuracy. In this application $p = 10$, thus, $p \times N \times l = 3 \times 10^3$; the size of the matrix holding time data used by the STD is $210 \times 40 = 8.4 \times 10^3$. This is about three times as many data for the STD as compared to the LSMBT. However, by increasing the problem size to reduce the noise effect, computer storage and numerical errors also increase.

The comparison of results between the LSMBT and the STD is given in Table 4. The modal frequencies are recovered exactly in the case of the LSMBT. More important, the damping ratio estimates are comparable for the two methods even though the LSMBT uses lesser data.

Conclusions

The least-squares moving-block technique (LSMBT) has been shown to yield comparable results to the Ibrahim sparse time domain algorithm. However, the implementation of the LSMBT has the following advantages:

1) A relatively large number of time signals are not required. In fact, the method can be applied starting from a single time signal. This is of interest because experimental data are not always available at a large number of recording stations.

2) There is no need to postprocess the results to separate computational and physical modes.

3) There is no need for an initial guess of system order.

4) Control over the accuracy is achieved by increasing or decreasing the number of blocks.

5) Computer storage and CPU time are reduced.

The implementation advantages and results of this new method demonstrate its strong capability as a tool for modal identification.

References

[1]Hammond, C. E., and Dogget, R. V., Jr., "Determination of Subcritical Damping by Moving-Block/Randomdec Applications," NASA SP-415, Oct. 1975.

[2]Ibrahim, S. R., "An Approach for Reducing Computational Requirements in Modal Identification," *AIAA Journal*, Vol. 24, No. 10, 1986, pp. 1725–1727.

[3]Kauffman, R. R., "Application of the Ibrahim Time Domain Algorithm to Spacecraft Transient Responses," AIAA Paper 84-0946, May 1984.

[4]Juang, J.-N., and Pappa, R. S., "An Eigensystem Realization Algorithm for Modal Parameter Identification and Model Reduction," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 5, 1985, pp. 620–627.

[5]Ljung, L., and Soderstrom, T., *Theory and Practice of Recursive Identification*, MIT Press, Cambridge, MA, 1983.

[6]Omar Amrani, A., and Du Val, R., "Parameter Identification of Aeroelastic Modes of Rotary Wings from Transient Time Histories," *Journal of Guidance, Control, and Dynamics* (to be published).

Contribution of Zonal Harmonics to Gravitational Moment

Carlos M. Roithmayr*
NASA Johnson Space Center, Houston, Texas 77058

Introduction

THE gravitational moment about the mass center of a body in orbit about a celestial body has an important effect on the orientation of the orbiting body. The more misshapen the celestial body, and the less uniform its mass distribution, the more involved is the calculation of the gravitational moment (and force) it exerts. Situations in which it might be important to calculate accurately the gravitational moment include the design of spacecraft for expeditions to asteroids, comets, and the moons of Mars.

In Ref. 1, a method for obtaining a vector-dyadic expression for the moment exerted about a small body's mass center by an oblate spheroid was set forth. The derivation of that expression made use of a gravitational potential written in terms of the zonal harmonic of the second degree. When gravitational potentials containing zonal harmonics of degree 2 or greater are considered, each zonal harmonic makes a contribution to the gravitational moment.

What follows is a vector-dyadic expression for the contribution of a zonal harmonic of degree n to the gravitational moment, produced by a body, about the mass center of a small body. As is the case with all vector-dyadic expressions, this result is basis-independent—that is, the vectors and dyadics can be expressed in any convenient vector basis.

The equation that follows is recursive: The contribution to the gravitational moment from the zonal harmonic of degree

Received April 6, 1989; revision received June 21, 1989.

*Aerospace Engineer, Guidance and Navigation Branch.

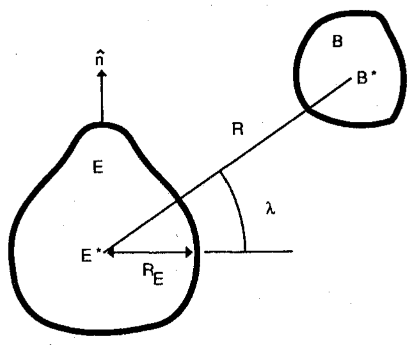

Fig. 1 Body B in the presence of body E.

n is a function of the moment contributions from the zonal harmonics of degree $n-1$ and $n-2$. The equation contains Legendre polynomials and derivatives of Legendre polynomials of degree $n-1$ and $n-2$. The Legendre polynomials, as well as their derivatives, can themselves be generated by means of recursion formulas.

As an example, the contribution to the gravitational moment from the zonal harmonic of degree 2 is worked out. The contribution of the zonal harmonic of degree 3 is also given.

Assertion

Figure 1 shows a small body B in the presence of an axisymmetric body E. The distance between B^*, the mass center of B, and E^*, the mass center of E, is assumed to exceed the greatest distance from B^* to any point of B. The system of gravitational forces exerted by E on B produces a moment $\boldsymbol{M}$ about B^*, and $\boldsymbol{M}$ can be written as

$$\boldsymbol{M} = \frac{3\mu}{R^3}\boldsymbol{r}\times\underline{I}\cdot\boldsymbol{r} + \sum_{n=2}^{\infty}\boldsymbol{M}_n \tag{1}$$

where $\boldsymbol{M}_n$ is the contribution of the zonal harmonic of degree n and can be obtained by using the recursion relation

$$\begin{aligned}\boldsymbol{M}_n = \frac{\mu J_n R_E^n}{nR^{n+3}}\Big\{\Big[(2n-1)\big[(n+2)P_{n-1}+3(\boldsymbol{r}\cdot\boldsymbol{n})P_{n-1}'\big]\\
-(2n-2)P_{n-2}'\Big](\boldsymbol{n}\times\underline{I}\cdot\boldsymbol{r}+\boldsymbol{r}\times\underline{I}\cdot\boldsymbol{n})\\
+\Big[(4n-4)\big[(n+1)P_{n-2}+(\boldsymbol{r}\cdot\boldsymbol{n})P_{n-2}'\big]\\
-(2n-1)(\boldsymbol{r}\cdot\boldsymbol{n})\big[(4n+8)P_{n-1}+4(\boldsymbol{r}\cdot\boldsymbol{n})P_{n-1}'\big]\Big]\boldsymbol{r}\times\underline{I}\cdot\boldsymbol{r}\\
-\big[2(2n-1)P_{n-1}'\big]\boldsymbol{n}\times\underline{I}\cdot\boldsymbol{n}\Big\} + \frac{2n-1}{n}\frac{J_n}{J_{n-1}}\frac{R_E}{R}(\boldsymbol{r}\cdot\boldsymbol{n})\boldsymbol{M}_{n-1}\\
-\frac{n-1}{n}\frac{J_n}{J_{n-2}}\left(\frac{R_E}{R}\right)^2\boldsymbol{M}_{n-2}\end{aligned} \tag{2}$$

where μ is the gravitational parameter of E, J_n is the zonal harmonic coefficient of degree n, R_E is the mean equatorial radius of E, R is the distance from E^* to B^*, P_n is the Legendre polynomial of degree n and argument S_λ, S_λ is the sine of λ, the latitude of B^* ($S_\lambda = \boldsymbol{r}\cdot\boldsymbol{n}$), P_n' is the first derivative, with respect to its argument, of P_n, $\boldsymbol{r}$ is the unit position vector from E^* to B^*, $\boldsymbol{n}$ is the unit vector in the direction of the axis of symmetry of E, and $\underline{I}$ is the inertia dyadic of B relative to B^*.

Derivation

The gravitational forces exerted by a body E on a small body produce a moment $\boldsymbol{M}$ about the mass center B^* of B. $\boldsymbol{M}$ is given approximately by Eq. (2.18.1) of Ref. 2

$$\boldsymbol{M} = -\underline{I}\overset{\times}{\cdot}\nabla\nabla V(\boldsymbol{R}) \tag{3}$$

where $\boldsymbol{R}$ is the position vector from E^* to B^* and ∇ denotes differentiation with respect to the vector $\boldsymbol{R}$. Section 2.9 of Ref. 2 contains a thorough explanation of how one differentiates with respect to a vector. The definition of the cross-dot product, $\overset{\times}{\cdot}$, appears on p. 156 of Ref. 2. The gravitational potential of E is symbolized by V.

Equation (2.13.14) in Ref. 2 deals with the gravitational potential of an axisymmetric body and contains an infinite series of zonal harmonics. For a particle of unit mass coincident with B^*,

$$V = \frac{\mu}{R}\left[1-\sum_{n=2}^{\infty}\left(\frac{R_E}{R}\right)^n J_n P_n(S_\lambda)\right] \tag{4}$$

where S_λ, the argument of P_n, is equal to the sine of the geographic latitude of B^*.

$\boldsymbol{M}_n$, the contribution of the nth zonal harmonic to the gravitational moment, is defined as

$$\boldsymbol{M}_n \equiv \mu J_n R_E^n\left\{\underline{I}\overset{\times}{\cdot}\nabla\nabla\left[\frac{P_n(S_\lambda)}{R^{n+1}}\right]\right\} \tag{5}$$

Reference 1 shows that $-\underline{I}\overset{\times}{\cdot}\nabla\nabla(\mu/R) = (3\mu/R^3)\boldsymbol{r}\times\underline{I}\cdot\boldsymbol{r}$. Hence, Eq. (3) can be rewritten as

$$\boldsymbol{M} = \frac{3\mu}{R^3}\boldsymbol{r}\times\underline{I}\cdot\boldsymbol{r} + \sum_{n=2}^{\infty}\boldsymbol{M}_n \tag{6}$$

in agreement with Eq. (1).

The Legendre polynomial of degree n, $P_n(x)$, is expressed recursively in Eq. (8.71) of Ref. 3 in terms of Legendre polynomials of degree $n-1$, $n-2$, and their argument, x, for $n \geq 2$, as follows:

$$P_n(x) = \frac{1}{n}\left[(2n-1)xP_{n-1}(x)-(n-1)P_{n-2}(x)\right] \tag{7}$$

Equation (7) can also be produced with $m=0$ in formula I of Table 1 in Ref. 4. Introducing this recursion relation for the Legendre polynomials will ultimately allow us to find a recursive expression for $\boldsymbol{M}_n$.

Equation (5) requires that a dyadic be formed by differentiating a scalar twice with respect to $\boldsymbol{R}$. We define $\rho_n(S_\lambda)$, a scalar of degree n and argument S_λ, as

$$\rho_n(S_\lambda) \equiv \frac{P_n(S_\lambda)}{R^{n+1}} \tag{8}$$

and $\underline{D}_n(S_\lambda)$, a dyadic of degree n and argument S_λ, as

$$\underline{D}_n(S_\lambda) \equiv \nabla\nabla\rho_n(S_\lambda) \tag{9}$$

A recursive relation for ρ_n can be obtained by substituting from Eq. (7) into (8)

$$\begin{aligned}\rho_n(S_\lambda) &= \frac{1}{nR^{n+1}}\left[(2n-1)S_\lambda P_{n-1}(S_\lambda)-(n-1)P_{n-2}(S_\lambda)\right]\\
&= \left[\frac{(2n-1)S_\lambda}{nR}\frac{P_{n-1}(S_\lambda)}{R^n}-\frac{n-1}{nR^2}\frac{P_{n-2}(S_\lambda)}{R^{n-1}}\right]\\
&= \left[\frac{(2n-1)S_\lambda}{nR}\rho_{n-1}(S_\lambda)-\frac{n-1}{nR^2}\rho_{n-2}(S_\lambda)\right]\end{aligned} \tag{10}$$

The first derivative of ρ_n with respect to $\boldsymbol{R}$ yields a vector $\nabla\rho_n$

$$\begin{aligned}\nabla\rho_n = \frac{2n-1}{n}\left(\frac{\rho_{n-1}}{R}\nabla S_\lambda+\frac{S_\lambda}{R}\nabla\rho_{n-1}-\frac{S_\lambda\rho_{n-1}}{R^3}\boldsymbol{R}\right)\\
-\frac{n-1}{n}\left(\frac{\nabla\rho_{n-2}}{R^2}-\frac{2\rho_{n-2}}{R^4}\boldsymbol{R}\right)\end{aligned} \tag{11}$$

The second derivative of ρ_n with respect to $\boldsymbol{R}$ yields a recursion relation for a symmetric dyadic $\underline{D}_n$

$$\underline{D}_n = \nabla\nabla\rho_n = \frac{2n-1}{n}\left(\frac{\rho_{n-1}}{R}\nabla\nabla S_\lambda + \frac{\nabla S_\lambda \nabla\rho_{n-1}}{R} - \frac{\rho_{n-1}\nabla S_\lambda \boldsymbol{R}}{R^3} + \frac{\nabla\rho_{n-1}\nabla S_\lambda}{R} + \frac{S_\lambda}{R}\underline{D}_{n-1} - \frac{S_\lambda \nabla\rho_{n-1}\boldsymbol{R}}{R^3} - \frac{\rho_{n-1}\boldsymbol{R}\nabla S_\lambda}{R^3} - \frac{S_\lambda \boldsymbol{R}\nabla\rho_{n-1}}{R^3} + \frac{3S_\lambda\rho_{n-1}\boldsymbol{RR}}{R^5} - \frac{S_\lambda\rho_{n-1}}{R^3}\underline{U}\right) - \frac{n-1}{n}\left(\frac{\underline{D}_{n-2}}{R^2} - \frac{2\nabla\rho_{n-2}\boldsymbol{R}}{R^4} - \frac{2\boldsymbol{R}\nabla\rho_{n-2}}{R^4} + \frac{8\rho_{n-2}\boldsymbol{RR}}{R^6} - \frac{2\rho_{n-2}}{R^4}\underline{U}\right) \tag{12}$$

where $\underline{U}$ is the unit dyadic, and the symmetric dyadics $\nabla\nabla\rho_{n-1}$ and $\nabla\nabla\rho_{n-2}$ have been renamed $\underline{D}_{n-1}$ and $\underline{D}_{n-2}$, respectively, by appealing to the definition in Eq. (9).

By substituting from Eqs. (8) and (9) into Eq. (5) and rearranging the result so that the cross-dot product of $\underline{I}$ and $\underline{D}_n$ is on the left-hand side, one obtains

$$\underline{I} \overset{\times}{\cdot} \underline{D}_n = \frac{\boldsymbol{M}_n}{\mu J_n R_E^n} \tag{13}$$

Cross-Dot Products

We now perform the cross-dot product with dyadics $\underline{I}$ and $\underline{D}_n$, making use of the right-hand side of Eq. (12) and the cross-dot identity $\underline{I} \overset{\times}{\cdot} \underline{U} = \boldsymbol{0}$, which is set forth in Eq. (19) of Ref. 1, obtaining

$$\underline{I} \overset{\times}{\cdot} \underline{D}_n = \frac{2n-1}{n}\left[\frac{\rho_{n-1}}{R}\underline{I} \overset{\times}{\cdot} \nabla\nabla S_\lambda + \frac{1}{R}\underline{I} \overset{\times}{\cdot} (\nabla S_\lambda \nabla\rho_{n-1} + \nabla\rho_{n-1}\nabla S_\lambda) + \frac{S_\lambda}{R}\underline{I} \overset{\times}{\cdot} \underline{D}_{n-1} - \frac{\rho_{n-1}}{R^3}\underline{I} \overset{\times}{\cdot} (\nabla S_\lambda \boldsymbol{R} + \boldsymbol{R}\nabla S_\lambda) - \frac{S_\lambda}{R^3}\underline{I} \overset{\times}{\cdot} (\nabla\rho_{n-1}\boldsymbol{R} + \boldsymbol{R}\nabla\rho_{n-1}) + \frac{3S_\lambda\rho_{n-1}}{R^5}\underline{I} \overset{\times}{\cdot} \boldsymbol{RR} - \boldsymbol{0}\right] - \frac{n-1}{n}\left[\frac{1}{R^2}\underline{I} \overset{\times}{\cdot} \underline{D}_{n-2} - \frac{2}{R^4}\underline{I} \overset{\times}{\cdot} (\nabla\rho_{n-2}\boldsymbol{R} + \boldsymbol{R}\nabla\rho_{n-2}) + \frac{8\rho_{n-2}}{R^6}\underline{I} \overset{\times}{\cdot} \boldsymbol{RR} - \boldsymbol{0}\right] \tag{14}$$

To carry out the cross-dot products with $\underline{I}$ and the other dyadics on the right side of Eq. (14), we will express these dyadics in terms of $\boldsymbol{R}$ and $\boldsymbol{n}$, a unit vector parallel to the axis of symmetry of E.

The sine of λ can be expressed as $\sin\lambda = (\boldsymbol{R}\cdot\boldsymbol{n})/R = \boldsymbol{r}\cdot\boldsymbol{n}$ so that the first derivative of S_λ with respect to $\boldsymbol{R}$ is

$$\nabla S_\lambda = \nabla\frac{\boldsymbol{R}\cdot\boldsymbol{n}}{R} = \frac{\boldsymbol{n}}{R} - \frac{(\boldsymbol{R}\cdot\boldsymbol{n})\boldsymbol{R}}{R^3} \tag{15}$$

and the second derivative of S_λ with respect to $\boldsymbol{R}$ is

$$\nabla\nabla S_\lambda = \frac{3(\boldsymbol{R}\cdot\boldsymbol{n})\boldsymbol{RR}}{R^5} - \frac{1}{R^3}\left[\boldsymbol{nR} + \boldsymbol{Rn} + (\boldsymbol{R}\cdot\boldsymbol{n})\underline{U}\right] \tag{16}$$

Equation (20) of Ref. 1 is a derivation of a cross-dot identity that will be used repeatedly throughout the sequel: For any dyad $\boldsymbol{uv}$ composed of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, it is shown that $\underline{I} \overset{\times}{\cdot} \boldsymbol{uv} = -\boldsymbol{u}\times\underline{I}\cdot\boldsymbol{v}$. By making use of this identity and Eq. (16), one can write

$$\underline{I} \overset{\times}{\cdot} \nabla\nabla S_\lambda = \frac{1}{R^2}(\boldsymbol{n}\times\underline{I}\cdot\boldsymbol{r} + \boldsymbol{r}\times\underline{I}\cdot\boldsymbol{n}) + \boldsymbol{0} - \frac{3(\boldsymbol{R}\cdot\boldsymbol{n})}{R^3}\boldsymbol{r}\times\underline{I}\cdot\boldsymbol{r} \tag{17}$$

where $\boldsymbol{r}$ is a unit vector in the direction of $\boldsymbol{R}$. By recalling the definition in Eq. (8), one can evaluate the first cross-dot product on the right side of Eq. (14):

$$\frac{\rho_{n-1}}{R}\underline{I} \overset{\times}{\cdot} \nabla\nabla S_\lambda = \frac{P_{n-1}}{R^{n+3}}\left[\boldsymbol{n}\times\underline{I}\cdot\boldsymbol{r} + \boldsymbol{r}\times\underline{I}\cdot\boldsymbol{n} - 3(\boldsymbol{r}\cdot\boldsymbol{n})\boldsymbol{r}\times\underline{I}\cdot\boldsymbol{r}\right] \tag{18}$$

The second cross-dot product on the right side of Eq. (14) contains the quantity $\nabla\rho_{n-1}$. Temporarily allow x to be the argument of ρ_{n-1} [see Eq. (8)] and write the derivative of ρ_{n-1} with respect to $\boldsymbol{R}$ as

$$\nabla\rho_{n-1}(x) = \frac{1}{R^n}\frac{\mathrm{d}}{\mathrm{d}x}P_{n-1}(x)\nabla x - \frac{nP_{n-1}(x)}{R^{n+2}}\boldsymbol{R} \tag{19}$$

The first derivative of a Legendre polynomial P_n with respect to its argument is often denoted by P_n'. For $n \geq 2$, a useful recurrence formula for P_n' can be found in problem 8-9 on p. 393 of Ref. 3, or in formula I of Table 1 in Ref. 4 (with $m = 1$):

$$P_n'(x) = \frac{1}{n-1}\left[(2n-1)xP_{n-1}'(x) - nP_{n-2}'(x)\right] \tag{20}$$

Since the argument of ρ_{n-1} is known to be S_λ, we can make use of Eq. (15) to rewrite Eq. (19) as

$$\nabla\rho_{n-1} = \frac{P_{n-1}'}{R^n}\left[\frac{\boldsymbol{n}}{R} - \frac{(\boldsymbol{R}\cdot\boldsymbol{n})\boldsymbol{R}}{R^3}\right] - \frac{nP_{n-1}}{R^{n+2}}\boldsymbol{R} \tag{21}$$

The sum of two dyadics, formed by juxtaposing the vectors ∇S_λ and $\nabla\rho_{n-1}$ in opposite order, yields the symmetric dyadic

$$\nabla S_\lambda\nabla\rho_{n-1} + \nabla\rho_{n-1}\nabla S_\lambda = \frac{2P_{n-1}'}{R^n}\left[\frac{\boldsymbol{nn}}{R^2} - \frac{(\boldsymbol{R}\cdot\boldsymbol{n})}{R^4}(\boldsymbol{Rn} + \boldsymbol{nR}) + \frac{(\boldsymbol{R}\cdot\boldsymbol{n})^2}{R^6}\boldsymbol{RR}\right] - \frac{nP_{n-1}}{R^{n+2}}\left[\frac{1}{R}(\boldsymbol{Rn} + \boldsymbol{nR}) - \frac{2(\boldsymbol{R}\cdot\boldsymbol{n})}{R^3}\boldsymbol{RR}\right] \tag{22}$$

Consequently, the second cross-dot product on the right side of Eq. (14) can be expressed as

$$\frac{1}{R}\underline{I} \overset{\times}{\cdot} (\nabla S_\lambda\nabla\rho_{n-1} + \nabla\rho_{n-1}\nabla S_\lambda) = \frac{2P_{n-1}'}{R^{n+3}}\left[(\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{r}\times\underline{I}\cdot\boldsymbol{n} + \boldsymbol{n}\times\underline{I}\cdot\boldsymbol{r}) - \boldsymbol{n}\times\underline{I}\cdot\boldsymbol{n} - (\boldsymbol{r}\cdot\boldsymbol{n})^2\boldsymbol{r}\times\underline{I}\cdot\boldsymbol{r}\right] - \frac{nP_{n-1}}{R^{n+3}}\left[2(\boldsymbol{r}\cdot\boldsymbol{n})\boldsymbol{r}\times\underline{I}\cdot\boldsymbol{r} - \boldsymbol{n}\times\underline{I}\cdot\boldsymbol{r} - \boldsymbol{r}\times\underline{I}\cdot\boldsymbol{n}\right] \tag{23}$$

The third cross-dot product on the right side of Eq. (14) can be expressed in terms of $\boldsymbol{M}_{n-1}$. Replacing n with $n-1$ in Eq. (13), we get

$$\frac{S_\lambda}{R}\underline{I} \overset{\times}{\cdot} \underline{D}_{n-1} = \frac{\boldsymbol{r}\cdot\boldsymbol{n}}{R}\frac{\boldsymbol{M}_{n-1}}{\mu J_{n-1}R_E^{n-1}} \tag{24}$$

The dyadic required for the fourth cross-dot product on the right side of Eq. (14) is easily constructed by using Eq. (15):

$$\nabla S_\lambda\boldsymbol{R} + \boldsymbol{R}\nabla S_\lambda = \boldsymbol{nr} + \boldsymbol{rn} - 2(\boldsymbol{r}\cdot\boldsymbol{n})\boldsymbol{rr} \tag{25}$$

Thus,

$$\frac{\rho_{n-1}}{R^3}\underline{I}\overset{\times}{\cdot}(\nabla S_\lambda R + R\nabla S_\lambda)$$
$$= \frac{P_{n-1}}{R^{n+3}}\left[2(r\cdot n)r\times\underline{I}\cdot r - n\times\underline{I}\cdot r - r\times\underline{I}\cdot n\right] \tag{26}$$

The dyadic required for the fifth cross-dot product on the right side of Eq. (14) also can be constructed rather easily by employing Eq. (21), which yields

$$\nabla\rho_{n-1}R + R\nabla\rho_{n-1} = \frac{P'_{n-1}}{R^n}\left[nr + rn - 2(r\cdot n)rr\right]$$
$$-\frac{2nP_{n-1}}{R^n}rr \tag{27}$$

so that

$$\frac{S_\lambda}{R^3}\underline{I}\overset{\times}{\cdot}(\nabla\rho_{n-1}R + R\nabla\rho_{n-1})$$
$$= \frac{(r\cdot n)P'_{n-1}}{R^{n+3}}\left[2(r\cdot n)r\times\underline{I}\cdot r - n\times\underline{I}\cdot r - r\times\underline{I}\cdot n\right]$$
$$+\frac{2n(r\cdot n)P_{n-1}}{R^{n+3}}r\times\underline{I}\cdot r \tag{28}$$

The sixth cross-dot multiplication that must be performed in order to obtain $\underline{I}\overset{\times}{\cdot}\underline{D}_n$ is one of the easiest to carry out. That is,

$$\frac{3S_\lambda\rho_{n-1}}{R^5}\underline{I}\overset{\times}{\cdot}RR = -\frac{3(r\cdot n)P_{n-1}}{R^{n+3}}r\times\underline{I}\cdot r \tag{29}$$

The seventh cross-dot product on the right side of Eq. (14) can be expressed in terms of M_{n-2}. Replacing n with $n-2$ in Eq. (13), we get

$$\frac{1}{R^2}\underline{I}\overset{\times}{\cdot}\underline{D}_{n-2} = \frac{1}{R^2}\frac{M_{n-2}}{\mu J_{n-2}R_E^{n-2}} \tag{30}$$

The dyadic required for the eighth cross-dot product is similar to that needed for the fifth cross-dot product

$$\nabla\rho_{n-2}R + R\nabla\rho_{n-2} = \frac{P'_{n-2}}{R^{n-1}}\left[nr + rn - 2(r\cdot n)rr\right]$$
$$-\frac{2(n-1)P_{n-2}}{R^{n-1}}rr \tag{31}$$

so that

$$\frac{2}{R^4}\underline{I}\overset{\times}{\cdot}(\nabla\rho_{n-2}R + R\nabla\rho_{n-2})$$
$$= \frac{2P'_{n-2}}{R^{n+3}}\left[2(r\cdot n)r\times\underline{I}\cdot r - n\times\underline{I}\cdot r - r\times\underline{I}\cdot n\right]$$
$$+\frac{4(n-1)P_{n-2}}{R^{n+3}}r\times\underline{I}\cdot r \tag{32}$$

The final cross-dot product to be evaluated is simply

$$\frac{8\rho_{n-2}}{R^6}\underline{I}\overset{\times}{\cdot}RR = -\frac{8P_{n-2}}{R^{n+3}}r\times\underline{I}\cdot r \tag{33}$$

Substituting from Eqs. (18), (23), (24), (26), (28-30), (32), and (33) into Eq. (14) and then into Eq. (13) leads to Eq. (2), which is a recursive vector-dyadic expression for the contribution of the nth zonal harmonic to the gravitational moment.

Examples

To demonstrate the use of Eq. (2), we will use it to obtain M_2, the contribution to the gravitational moment from the zonal harmonic of degree 2.

The two required Legendre polynomials are $P_1(S_\lambda) = S_\lambda = r\cdot n$ and $P_0(S_\lambda) = 1$. Legendre polynomials of degree greater than or equal to 2 can be obtained recursively by using Eq. (7). The two derivatives of Legendre polynomials that will be needed are $P'_1(S_\lambda) = 1$ and $P'_0(S_\lambda) = 0$. Derivatives with respect to the argument of Legendre polynomials can be generated with the recursion formula [Eq. (20)] for $n \geq 2$.

Equation (2) also requires knowledge of M_0 and M_1 in order to produce M_2. Equation (13) is helpful in developing expressions for M_0 and M_1.

The Legendre polynomial of degree zero is equal to 1, regardless of its argument, and the scalar ρ_0 [see Eq. (8)] is

$$\rho_0 = \frac{P_0}{R} = \frac{1}{R} \tag{34}$$

The dyadic of degree zero [see Eq. (9)] is then

$$\underline{D}_0 = \nabla\nabla\rho_0 = \nabla\nabla\left(\frac{1}{R}\right) = \nabla\left(-\frac{R}{R^3}\right)$$
$$= \left(\frac{3}{R^5}\right)RR - \left(\frac{1}{R^3}\right)\underline{U} \tag{35}$$

Equation (13) tells us that

$$M_0 = -\frac{3\mu J_0}{R^3}r\times\underline{I}\cdot r \tag{36}$$

J_0 is an undefined constant, but the coefficient of M_0 in Eq. (2) contains J_0 in the denominator. Hence, a numerical value of J_0 is not required for constructing M_2.

A similar process leads to M_1. The value of the Legendre polynomial of degree 1 is identical to the argument, so the scalar ρ_1 is

$$\rho_1 = \frac{P_1}{R^2} = \frac{(R\cdot n)}{R^3} \tag{37}$$

The dyadic of degree 1 and argument S_λ is

$$\underline{D}_1 = \nabla\nabla\rho_1 = \nabla\nabla\frac{(R\cdot n)}{R^3} = \nabla\left[\frac{n}{R^3} - \frac{3(R\cdot n)}{R^5}R\right]$$
$$= \frac{15(R\cdot n)}{R^7}RR - \frac{3}{R^5}\left[Rn + nR + (R\cdot n)\underline{U}\right] \tag{38}$$

so that

$$M_1 = \frac{3\mu J_1 R_E}{R^4}\left[n\times\underline{I}\cdot r + r\times\underline{I}\cdot n - 5(r\cdot n)r\times\underline{I}\cdot r\right] \tag{39}$$

Like J_0, the constant J_1 is undefined and unneeded for the purpose of obtaining M_2. Note that M_0 and M_1 do not represent contributions to the gravitational moment, but are required to begin the process of recursion that will generate moment contributions beginning with M_2.

By substituting from Eqs. (36) and (39) into Eq. (2), we arrive at the following result with $n = 2$:

$$M_2 = \frac{\mu J_2 R_E^2}{2R^5}\Big\{\left[30(r\cdot n)\right](n\times\underline{I}\cdot r + r\times\underline{I}\cdot n)$$
$$+\left[15 - 105(r\cdot n)^2\right]r\times\underline{I}\cdot r - 6n\times\underline{I}\cdot n\Big\} \tag{40}$$

If Eq. (1) of Ref. 1 is expressed as $M = (3\mu/R^3)r\times\underline{I}\cdot r + M_2$, it can be seen that M_2 from Ref. 1 is identical to the foregoing Eq. (40).

The contribution M_3 can be obtained in a similar manner, using the values of $P_2(S_\lambda)$, $P_1(S_\lambda)$, $P_2'(S_\lambda)$, $P_1'(S_\lambda)$, M_2, and M_1:

$$M_3 = \frac{\mu J_3 R_E^3}{6R^6}\Big\{\left[315(r\cdot n)^2-45\right](n\times \underline{I}\cdot r + r\times \underline{I}\cdot n) + \left[315(r\cdot n)-945(r\cdot n)^3\right] r\times \underline{I}\cdot r - 90(r\cdot n) n\times \underline{I}\cdot n\Big\} \quad (41)$$

Conclusions

A recursive vector-dyadic expression for the contribution of a zonal harmonic of degree n to the gravitational moment about the mass center of a small body can be obtained by a procedure that involves differentiating a celestial body's gravitational potential twice with respect to a vector. The recursive property of the result is a consequence of taking advantage of a recursion relation for Legendre polynomials that appear in the gravitational potential. When a celestial body's gravitational potential includes zonal harmonics, the preceding vector-dyadic expression is useful for calculating their contributions to the gravitational moment. The contribution of the zonal harmonic of degree 2 is consistent with the gravitational moment exerted by an oblate spheroid.

References

[1]Roithmayr, C. M., "Gravitational Moment Exerted on a Small Body by an Oblate Body," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 3, 1989, pp. 441–444.

[2]Kane, T. R., Likins, P. W., and Levinson, D. A., *Spacecraft Dynamics,* McGraw-Hill, New York, 1983, Chap. 2.

[3]Battin, R. H., *An Introduction to the Mathematics and Methods of Astrodynamics,* AIAA, New York, 1987, p. 393.

[4]Lundberg, J. B., and Schutz, B. E., "Recursion Formulas of Legendre Functions for Use with Nonsingular Geopotential Models," *Journal of Guidance, Control, and Dynamics,* Vol. 11, No. 1, 1988, pp. 32–38.

Satellite Relocation by Tether Deployment

Geoffrey A. Landis* and Frank J. Hracht†
NASA Lewis Research Center,
Cleveland, Ohio 44135

Introduction

ONE of the limitations of the working lifetime of a satellite is the expenditure of fuel required for changing and maintaining the orbit. In geosynchronous orbit, the orbit is subject to perturbations, primarily due to the gravitational effects of the sun and the moon. Fuel is also required for relocating a satellite, for example, if a synchronous satellite positioned over Indonesia is desired to be relocated over South America. We propose that a satellite can be repositioned in orbit without fuel expenditure by extending a mass on the end of a tether. The tether may also serve other purposes, including energy storage for eclipse power.

A tether is a long, flexible cable that connects one part of a satellite with another. Tethers have recently been well covered in the aerospace literature.[1-5] In the equilibrium configuration, as shown in Fig. 1, the tether is oriented radially outward, with a tension on the tether due to the gravitational gradient (or "tidal") force.

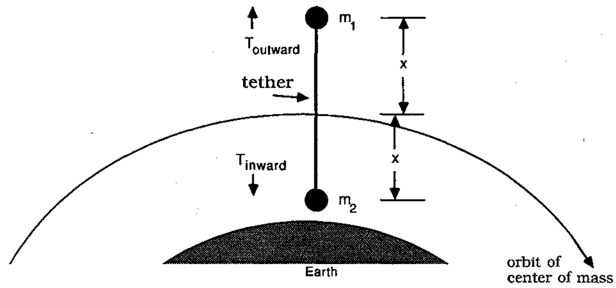

Fig. 1 Tether orbit and definitions.

Mass of the cable is an important figure. A figure of merit for material strength is the critical length L_c, the length of untapered cable that could be suspended in a 1-g gravitational field. One proposed cable material is Kevlar. For one common type of Kevlar, the critical length is 250 km.[6] The effective acceleration due to the gravity gradient a distance x from the center of mass (cm) is to first order

$$a_{\text{eff}} = 3g\, r_e^2 x / r_o^3 \quad (1)$$

where r_o is the orbital radius and r_e is the radius of the Earth. At geosynchronous Earth orbit (GEO), $3\, r_e^2/r_o^3 = 1.6\cdot 10^{-6}$ km^{-1}. The minimum mass m_t of untapered cable required to support an end mass m_o is thus to first order (for x in km):

$$m_t\,(\text{GEO}) = 1.6\cdot 10^{-6}\cdot m_o x^2/L_c \quad (2)$$

Thus, in GEO a 1000-km-long tether can easily be made much less massive than the satellite it supports.

Most analyses of tether orbits assume that the center of mass of a tethered satellite system remains in the original orbit,[1,2] i.e., that the angular velocity of the tethered satellite does not change as the tether is extended or retracted. We note that this is true only to the first-order approximation in tether length. Briefly, the mass that extends outward experiences an increase in centrifugal force that increases linearly with distance, but the mass that extends inward experiences an increase in gravity that increases faster than the linear increase. Thus, the center of mass of the orbit is pulled inward, and to conserve angular momentum, the angular velocity of the orbit increases.

Mathematical Analysis

In the following discussion, we shall assume a tether of negligible mass in circular orbit. The extension of the analysis to tethers of nonnegligible mass is straightforward.

Consider a satellite of mass m_i consisting of two pieces of masses m_1 and m_2 connected by a tether. For calculation simplicity, let $m_1 = m_2 = m_i/2$. The initial orbit is assumed to be circular, with an angular velocity ω_o and an initial orbital radius (measured from the Earth's center) r_o. With the tether at initial length zero, the orbit has initial angular momentum

$$L_i = m_i \omega_o r_o^2 \quad (3)$$

Received April 6, 1989; revision received June 15, 1989.

*National Research Council Resident Research Associate.

†Aerospace Engineer. Associate Fellow AIAA.